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# Temporal Lagrangian Decomposition of Model Predictive Control for Hybrid Systems

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*Abstract*— **Given a Model Predictive Control (MPC) problem for a hybrid system in the Mixed Logical Dynamical (MLD) framework, a temporal decomposition scheme is proposed that efficiently derives the control actions by performing Lagrangian decomposition on the prediction horizon. The algorithm translates the original optimal control problem into a temporal sequence of independent subproblems of smaller dimension. The solution of the Lagrangian problem yields a sequence of control actions for the full horizon that is approximate in nature due to the non-convexity of the hybrid optimal control problem formulation and the consequent duality gap. For cases, however, where the duality gap is sufficiently narrow, the approximate control law will yield almost the same closed-loop behavior as the one obtained from the original optimal controller, but with a considerably smaller computational burden. An example, for which a reduction of the computation time by an order of magnitude is achieved, illustrates the algorithm and confirms its effectiveness.**

## I. INTRODUCTION

Model Predictive Control (MPC) is a control methodology, in which the current control action is obtained by solving at each sampling instant an open-loop optimal control problem over a finite horizon using the current state of the plant as the initial state. The underlying optimization procedure yields an optimal control sequence that minimizes a given objective function. By only applying the first control move in this sequence and by recomputing the control sequence at the next sampling instant, a receding horizon policy is achieved. A major advantage of MPC is its ability to cope with hard constraints on manipulated variables, states and outputs. Furthermore the Mixed Logic Dynamical (MLD) framework can be embedded in MPC allowing one to use hybrid models given in the MLD form as prediction models [2].

In recent years, MPC has become a prominent control technique especially in the chemical process industry, because the large time constants typically involved in these applications allow for a reasonable computation time for the on-line optimization. For systems featuring smaller time constants, however, this may not always be possible because of the complexity of the calculations. One possibility is to compute an off-line solution to the optimization problem as a function of the state resulting in a lookup table. Then, at each time instant the determination of the optimal feedback law is performed by the simple evaluation of a function [1], [3]. However, in particular for hybrid systems, such an approach is computationally feasible only for problems with a low-dimensional state and a short prediction horizon.

Another approach to circumvent the complexity obstacle is to decompose the prediction horizon into a sequence of several sub-horizons. A natural choice to do so is to employ Lagrangian decomposition, which has been used successfully in a variety of circumstances for the purpose of translating an originally complex problem formulation into a simpler version consisting of a series of smaller disjoint elements. In the present paper, temporal decomposition is done by relaxing some of the state-update equations and adding them to the cost function as penalty terms multiplied with the Lagrangian (or dual) variables. The obtained disjoint problems constitute the primal subproblems and may be tackled independently, therefore reducing the computation time. The dual variables provide the link between the subproblems, and are determined by solving the dual problem.

It seems that such an approach based on temporal decomposition has not been attempted for MPC as yet. Nevertheless, in the context of planning and scheduling, temporal decomposition techniques have been used successfully, see e.g. [11].

As an illustrative example and case study, the optimal control problem of a four bus power system [9] is considered. In particular, the power system comprises a hybrid system featuring binary manipulated variables, a finite state machine and continuous dynamics. Compared to the original problem, the Lagrangian decomposition scheme shows favorable computation times which are reduced by an order of magnitude. Furthermore, in this example, the optimal control laws obtained from the decomposed and from the original problem coincide. In any case, as increasing the prediction interval in the context of generic hybrid systems quickly renders the related optimal control problems computationally intractable, it is to be expected that the gain in computational performance would be significant.

This paper is organized as follows. In Section II, the MLD framework is recapitulated and the optimal control problem is posed. Subsequently, Lagrangian decomposition is briefly summarized, and the proposed temporal decomposition scheme is presented in detail in Section III. Section IV features the numerical application to the voltage control problem exemplifying the efficacy of the aforementioned algorithm.

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## II. Optimal Control of Hybrid Systems

Consider a hybrid system modelled in the Mixed Logic Dynamical (MLD) framework [2]

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \quad (1a)$$
$$y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \quad (1b)$$
$$E_2\delta(k) + E_3z(k) \leq E_1u(k) + E_4x(k) + E_5\,, \quad (1c)$$

where $k \in \mathbb{N}$ denotes the discrete time-instant, and $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ denotes the states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ the inputs and $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ the outputs, with both real and binary components. Furthermore, $\delta \in \{0,1\}^{r_\ell}$ and $z \in \mathbb{R}^{r_c}$ represent binary and auxiliary continuous variables, respectively. These variables are introduced when translating propositional logic or piecewise affine (PWA) functions into linear inequalities.

Consider the constrained optimal control problem

$$\begin{aligned}\min_{U_N} J(x(0)) = \sum_{k=0}^{N-1} &||Q_1(u(k)-u_r)||_\infty + \\ &||Q_2(\delta(k)-\delta_r)||_\infty + ||Q_3(z(k)-z_r)||_\infty + \\ &||Q_4(x(k)-x_r)||_\infty + ||Q_5(y(k)-y_r)||_\infty \quad (2a)\\ \text{s. t.} \quad &(1)\\ &u_{min} \leq u(k) \leq u_{max} \quad (2b)\\ &x_{min} \leq x(k) \leq x_{max}\,, \quad (2c)\end{aligned}$$

where $N$ is the prediction horizon, $Q_1, ..., Q_5$ are positive semi-definite weighting matrices, $x_r$, $u_r$, $y_r$, $\delta_r$, $z_r$ are the respective references, $x(k)$ is the state predicted at time-step $k$ resulting from the input sequence $U_N = [u^T(0), ..., u^T(N-1)]^T$ starting from the state $x(0)$, $u_{min}$, $u_{max}$, $x_{min}$, $x_{max}$ are hard constraints on the inputs and the states respectively and $||\cdot||_\infty$ represents the infinity norm defined according to the usual relation $||v||_\infty = \max_i |v_i|$.

Let $U_N^*$ denote the optimal input sequence that minimizes $J(x(0))$. According to the receding horizon policy, only the first control move

$$u(0) = U^*(0) \quad (3)$$

is applied, and the whole optimization procedure is repeated at time-step $k+1$.

The MPC formulation can be rewritten [1] as a Mixed Integer Linear Program (MILP) by introducing the vector of slack variables $\epsilon = [\epsilon_0^u, ..., \epsilon_{N-1}^u, \epsilon_0^\delta, ..., \epsilon_{N-1}^\delta, \epsilon_0^z, ..., \epsilon_{N-1}^z, \epsilon_0^x, ..., \epsilon_{N-1}^x, \epsilon_0^y, ..., \epsilon_{N-1}^y]^T$, that satisfies for $k = 0, 1, ..., N-1$

$$1_m\epsilon_k^u \geq \pm\, Q_1(u(k)-u_r) \quad (4a)$$
$$1_{r_\ell}\epsilon_k^\delta \geq \pm\, Q_2(\delta(k)-\delta_r) \quad (4b)$$
$$1_{r_c}\epsilon_k^z \geq \pm\, Q_3(z(k)-z_r) \quad (4c)$$
$$1_n\epsilon_k^x \geq \pm\, Q_4(x(k)-x_r) \quad (4d)$$
$$1_p\epsilon_k^y \geq \pm\, Q_5(y(k)-y_r) \quad (4e)$$

where $1_i$ denotes a column vector of ones of length $i$, and $m = m_c + m_\ell$, $n = n_c + n_\ell$, $p = p_c + p_\ell$. It can be proven [5], that if the vector $\epsilon$ satisfies (4) and minimizes the cost function

$$J_\epsilon(x(0)) = 1_{5N}^T\epsilon \quad (5)$$

also solves the original problem (2), i.e. leads to the same optimum $J_\epsilon^*(x(0)) = J^*(x(0))$. Therefore, the original problem can be recast accordingly.

## III. Temporal Lagrangian Decomposition of Hybrid Systems

### A. Introduction to Lagrangian Decomposition

Consider the MILP primal problem in the

$$\min_{x_1,x_2} F = f_1^Tx_1 + f_2^Tx_2 \quad (6a)$$
$$\text{s. t. } A_1x_1 \leq b_1, \quad C_1x_1 = d_1 \quad (6b)$$
$$A_2x_2 \leq b_2, \quad C_2x_2 = d_2 \quad (6c)$$
$$G_1x_1 + G_2x_2 = d \quad (6d)$$

where $x_1$, $x_2$ may be both real and integer variables. Such a problem would be decomposable into two independent primal subproblems featuring the variables $x_1$, $x_2$ were it not for the presence of the equality constraint (6d). In this case, a Lagrangian decomposition scheme precisely yields such a decomposability property. More specifically, consider the problem

$$\begin{aligned}\min_{x_1,x_2} L(\lambda) = f_1^Tx_1 + f_2^Tx_2 + &\\ \lambda^T(G_1x_1 + G_2x_2 - d)& \quad (7a)\\ \text{s. t. (6b), (6c)}& \quad (7b)\end{aligned}$$

where $\lambda$ are the dual variables associated with the relaxed constraint (6d) and where $L(x_1, x_2)$ is the Lagrangian cost function. For a given $\lambda$, the relaxed problem (6) can be separated into the two disjoint subproblems

$$\min_{x_1} L_1(\lambda) = f_1^Tx_1 + \lambda^TG_1x_1 - \frac{\lambda^Td}{2} \quad (8a)$$
$$\text{s. t. } A_1x_1 \leq b_1, \quad C_1x_1 = d_1 \quad (8b)$$

and

$$\min_{x_2} L_2(\lambda) = f_2^Tx_2 + \lambda^TG_2x_2 - \frac{\lambda^Td}{2} \quad (9a)$$
$$\text{s. t. } A_2x_2 \leq b_2, \quad C_2x_2 = d_2 \quad (9b)$$

that can be solved independently of each other.

It is well known from mathematical programming theory [7] that the value of $L(\lambda)$ corresponding to an arbitrary value of $\lambda$ yields lower bounds to the optimum of the original primal problem. It is therefore of interest to obtain the best possible lower bound, that is to solve the problem

$$Z^* = \max_\lambda(\min_{x_1,x_2} L(\lambda)) \quad (10)$$

over the unconstrained dual variable state space. Problem (10) is referred to as the dual problem, and

$$Z = \min_{x_1,x_2} L(\lambda) \quad (11)$$

represents the dual function, which is known to be concave and non-differentiable [4]. If the original problem (6) had been convex, the solution to (10) would produce a value of (11) equal to the optimum of (6), i.e. $Z^* = F^*$, and correspondingly a set of dual variables $\lambda$ to which the optimal $x_1^*$ and $x_2^*$ would be associated. In the case of an MILP, the problem is non-convex. Therefore, the solution to (10) will yield a bound that is strictly inferior to the optimum of (6), i.e. $Z^* < F^*$, and correspondingly $x_1$ and $x_2$ that are in general infeasible with respect to (6d). The usual strategy in Lagrangian decomposition methods is then to solve the dual to near optimality by means of some iterative method to obtain good lower bounds. Subsequently, a heuristic method is devised to generate solutions satisfying (6d) that are therefore solutions to (6) and whose cost represents an upper bound for (6a) [6].

### B. Temporal Lagrangian Decomposition

The foregoing methodology turns out to be opportunely practicable when considering temporal decomposition of MPC problems: the prediction horizon inherent in the MILP formulation can be partitioned into independent subhorizons by relaxing the system state update equation (1a) as was done for (6d) above. The new problem, consisting of the minimization of the augmented cost function subject to the reduced set of constraints, may now be conveniently reformulated as a set of independent primal subproblems of reduced dimension. More specifically, let $I \in \{1, ..., N\}$ be the number of blocks in which one chooses to divide the prediction horizon and $i \in \{1, ..., I\}$ the block indices, where $i = 1$ denotes the first time block. $M_i \in \{0, ..., N-2\}$ shall be the position in the prediction horizon corresponding to the beginning of the $i$-th time block, where $x(M_1) = x(0)$. The length or duration of the $i$-th time block is $d_i = M_{i+1} - M_i$ for $i \in \{1, ..., I-1\}$, and $d_I = N - 1 - M_I$.

Dualizing (1a) yields the Lagrangian function

$$L(\lambda) = J_\epsilon(x(0)) + \sum_{i=2}^{I} \lambda_i^T (x(M_i) - g(M_i - 1)), \quad (12)$$

where it has been set

$$\begin{aligned} g(M_i - 1) =& Ax(M_i - 1) + B_1 u(M_i - 1) \\ &+ B_2 \delta(M_i - 1) + B_3 z(M_i - 1)) \end{aligned} \quad (13)$$

and where $\lambda_i$ is the vector of dual variables of the constraint (1a) that connects the $(i-1)$-th with the $i$-th block, and $\lambda = [\lambda_2^T, ..., \lambda_I^T]^T$. Let $\epsilon_i$ denote the slack variables contained in the $i$-th subproblem. Setting

$$J_\epsilon = 1_{5N}^T \epsilon = \sum_{i=1}^{I} 1_{5d_i}^T \epsilon_i \quad (14)$$

allows one to separate (12) into independent subproblems each referring to one time block, i.e.

$$L(\lambda) = L_1(\lambda) + \sum_{i=2}^{I-1} L_i(\lambda) + L_I(\lambda), \quad (15)$$

where

$$L_1(\lambda) = 1_{5d_1}^T \epsilon_1 - \lambda_2^T (g(M_2 - 1)) \quad (16)$$

$$L_i(\lambda) = 1_{5d_i}^T \epsilon_i - \lambda_{i+1}^T (g(M_{i+1} - 1)) + \lambda_i^T x(M_i) \quad (17)$$

and

$$L_I(\lambda) = 1_{5d_I}^T \epsilon_I + \lambda_I^T x(M_I). \quad (18)$$

### C. Formulation of the Primal Subproblems

The remaining system constraints in (1), and the constraints (4), (2b) and (2c) can also be partitioned into different subproblems pertaining to the respective time blocks, similarly as has been done for (8b) and (9b). The first subproblem corresponding to the first block is:

$$\min_{\epsilon_1} \; L_1(\lambda) \quad (19)$$

subject to the constraints (1), (4), (2b) and (2c) pertaining to the first time block. The subsequent subproblems with $i \in \{2, 3, ..., I\}$ relative to the following blocks are:

$$\min_{\epsilon_i, x(M_i)} \; L_i(\lambda) \quad (20)$$

subject to the constraints (1), (4), (2b) and (2c) pertaining to the $i$-th time block.

The foregoing subproblems are now uniquely linked by means of the dual variables $\lambda$ whose value depends on the solution of the dual problem.

### D. Formulation and Solution of the Dual Problem

The dual is a concave, PWA and non-differentiable function. An implicit and iterative method for the solution of the dual such as the subgradient method can be used.

In a subgradient algorithm, an initial value for the dual variables is chosen arbitrarily, and the associated relaxed problem is solved to compute a subgradient direction for the dual variables and thus to modify the multipliers in the computed direction. For the case of a dual function, a possible subgradient $D(\lambda^j)$ at iteration $j$ is the set of the relaxed constraints itself, that is

$$D(\lambda^j) = [D_2^T(\lambda^j), \ldots, D_I^T(\lambda^j)]^T \quad (21)$$

where it has been set

$$D_i(\lambda^j) = \; x^j(M_i) - g^j(M_i - 1) \quad (22)$$

in which all values correspond to the optimal solution of the primal obtained with $\lambda^j$, see (12). In general, for a given $\lambda^j$, $D_i(\lambda^j)$ is different from zero, and the corresponding primal set of variables $\nu^j$ with

$$\nu = [x^T \; u^T \; \delta^T \; z^T \; y^T]^T, \quad (23)$$

is infeasible.

In the following, the algorithm described in [10] will be applied. Given $\lambda^j$, the multipliers for the next iteration step are calculated by

$$\lambda^{j+1} = \lambda^j + \alpha^j D(\lambda^j), \quad (24)$$

with the step size $\alpha^j$ being generated by

$$\alpha^j = \frac{\beta^j(J^+ - Z^j)}{||D(\lambda^j)||^2}. \quad (25)$$

In (25), $Z^j$ is the value of the dual evaluated at $\lambda^j$ according to (11), and $J^+$ is the current best upper bound to the optimum of the original problem. In particular, $J^+$ corresponds to the best feasible primal solution obtained by means of some heuristic algorithm from the sequence of generally infeasible $\nu^j$. Furthermore, $\beta^j$ is a parameter initially set to a value $\beta^j \in (0, 2)$ which is subsequently reduced whenever $Z^j$ has failed to improve after a certain number of iterations. The iterative procedure is interrupted either after a certain number of fixed steps or whenever a certain accuracy is attained in the duality gap between the infeasible and feasible values, taking into account however that this will be limited by the intrinsic duality gap of the problem.

When solving the problem to optimality in a subsequent branch and bound algorithm, $J^+$ may be used as an upper bound. Alternatively, the feasible input vector may be implemented as a suboptimal input. Indeed, with the appropriate problem setup, it is known that even a suboptimal sequence of control inputs can ensure stability [2], [12]. In any case, the effectiveness of the employed heuristic algorithm is of critical importance to ensure satisfactory performance for the overall problem. Such an algorithm is highly application specific. Assuming that no constraints on states are present, one may simply collect all the inputs $u^j$ computed from the subproblems (19) and (20). Applying them to the MLD system leads to a feasible set of $x$, $y$, $\delta$ and $z$ by construction. If state constraints are given, however, the above procedure cannot guarantee that these constraints are respected for the entire horizon. In this case, a feasible input sequence must be built according to the given problem formulation; as an indication, one possibility might be to use the inputs corresponding to the first subhorizon block (or first few predictions steps) and then to re-optimize over the remaining horizon tail to enforce feasibility.

## IV. Illustrative Example

### A. Model Description

In [9], the authors have presented a novel emergency control scheme based on MPC that successfully stabilizes the voltage in an example power system with four buses. As shown in Fig. 1, this power system contains two generators, where the first one is an infinite bus or a large power system, whereas the second generator includes an internal controller regulating the voltage at bus 2 thus limiting the maximal amount of reactive power. Also the transformer incorporates an internal controller regulating the load voltage $V_{4m}$ within a dead-band around the voltage reference $V_{4m,ref}$. This controller is a finite state machine and allows changes of the tap position $n_T$ only every 30 s by one discrete step of $n_{step} = 0.02$. In addition, by setting $s_C$, discrete parts of the capacitor bank can be used to support the power system by producing reactive power close to the load. The distribution system is approximated using one load model aggregating the whole distribution system. Discrete parts of the load can be disconnected by using load shedding $s_L$.

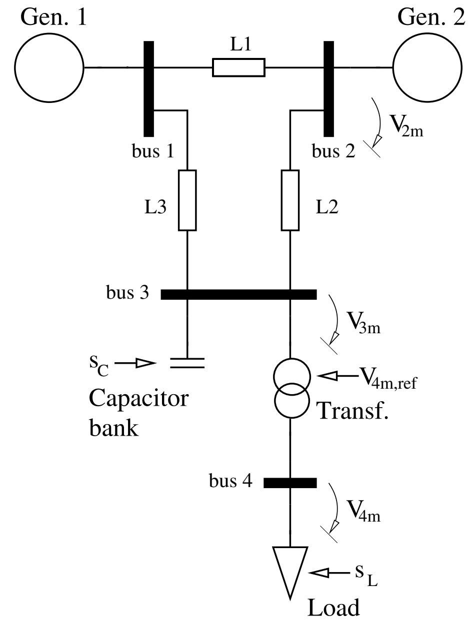


Fig. 1. Example power system with four buses.

Recapitulating [9], the MLD model is derived by performing the following approximations and simplifications. (i) The nonlinear continuous-time dynamics are approximated by PWA functions in the discrete-time domain using the sampling time $T_s = 30\,\mathrm{s}$, (ii) the load dynamics are simplified and modelled using only one load state $x_L$, (iii) the admissible sets of the input variables are reduced, and (iv) the tapping strategy $\Delta n_T \in \{0, n_{step}, -n_{step}\}$ is used as a system input rather than the reference voltage $V_{4m,ref}$. The resulting MLD model features the two-dimensional state vector $x = [x_L \; n_T]^T$ with $x_1 \in \mathbb{R}$ and $x_2 \in \{0.8, 0.82, 0.84, \dots, 1.2\}$, the four-dimensional input vector $u = [u_1 \; u_2 \; s_C \; s_L]^T$ with $u \in \{0,1\}^4$, where $u_1$ and $u_2$ encode in a binary fashion $\Delta n_T$, the outputs $y = [V_{2m} \; V_{3m} \; V_{4m}]$ with $y \in \mathbb{R}^3$ representing the bus voltages of main interest, 302 $z$-variables, 31 $\delta$-variables and 1660 inequality constraints.

### B. Control Problem Formulation

The control objectives are (i) to bring $V_{4m}$ as close to its reference value 1 as possible, (ii) to minimize switching transitions in the manipulated variables, and (iii) to fulfill the safety constraints on the bus voltages $V_{2m} \in [0.95, 1.05]$, $V_{3m} \in [0.9, 1.1]$ and $V_{4m} \in [0.9, 1.1]$. Modelling these constraints as soft constraints using the slack variables $s_i$, $i \in \{2,3,4\}$, yields

$$s_2(k) \geq 0.95 - V_{2m}(k) \tag{26a}$$
$$s_2(k) \geq V_{2m}(k) - 1.05 \tag{26b}$$
$$s_2(k) \geq 0\,, \tag{26c}$$

The slacks $s_3$ and $s_4$ are defined accordingly. Furthermore, let $s = [s_2\ s_3\ s_4]^T$.

Consider the optimal control problem

$$\min_{U_N} J(x(0), u(k-1)) = \sum_{k=0}^{N-1} \|y_3(k) - 1\| + \sum_{k=0}^{N-1} \Big( \|Q\Delta u(k)\|_\infty + \|Rs(k)\|_\infty \Big)$$

subject to the evolution of the MLD model (1) over the prediction horizon $N$ and the integrality constraints on $u(k)$, where $U_N = [u^T(0), u^T(1), \ldots, u^T(N-1)]^T$ denotes the sequence of control inputs, $\Delta u(k) = u(k) - u(k-1)$ the change in the manipulated variables, and $Q = \text{diag}(0.2, 0.2, 0.03, 0.1)$ and $R = \text{diag}(10, 10, 10)$ are the corresponding penalty matrices. This choice of the penalties will cause emergency actions including load shedding to be triggered only if refraining from such measures would result in a violation of the soft constraints. For more details on the four bus power system, the reader is referred to [8], [9].

### C. Temporal Lagrangian Decomposition

The proposed temporal Lagrangian decomposition scheme can be directly applied to the voltage control problem, as the model of the power system is given in MLD form and the optimal control problem is formulated using the $\infty$-norm. To test the performance of the presented scheme, a prediction horizon of $N = 6$ is adopted and decomposed into $I = 4$ separate blocks consisting of an initial block of length $d_1 = 3$ and of 3 successive blocks of length $d_2 = d_3 = d_4 = 1$. This choice largely depends on computational experiments; indeed, intuitively it can be assumed that employing a comparatively long initial block is beneficial for an application to MPC problems, where the quality of the first control move is essential for the overall performance of the control scheme.

It should be noticed that for the on-line solution of the MPC problem, the sampling interval $T_s = 30s$ imposes the maximum computation time available. Because of this stringent condition, there is no time available to perform any iterations in the dual variables on-line. Therefore, the dual variables are pre-computed off-line for a given characteristic state and are kept at the constant value $\bar{\lambda}$. For this particular example, it turns out that the optimal dual variables remain rather unaffected by variations in the state. Therefore, $\bar{\lambda}$ yields good results.

At each time step, the infeasible set of variables corresponding to $\bar{\lambda}$ is evaluated, and starting from this a feasible solution is built. As the power system does not feature any (hard) constraints on states, one may – as previously mentioned – simply retain the calculated input variables and build a feasible solution set and cost value. Other approaches are possible, and indeed, a better feasible solution may be obtained by maintaining only the first few inputs of the sequence and re-optimizing over the remaining part of the horizon. According to the receding horizon policy of MPC, from this feasible input sequence only the first move is applied, and the same operation is repeated at the next time instant.

| | Full horizon | Lagrangian scheme |
|---|---|---|
| Min. time [s] | 100 | 16.8 |
| Max. time [s] | 486 | 25.2 |
| Avg. time [s] | 276 | 18.6 |
| Min. $\gamma$ [%] | | 6.3 |
| Max. $\gamma$ [%] | | 12.3 |
| Avg. $\gamma$ [%] | | 9.1 |

TABLE I

COMPARISON OF THE COMPUTATION TIMES FOR THE FULL PROBLEM AT EACH SAMPLING INTERVAL AND THE LAGRANGIAN DECOMPOSITION SCHEME WITH THE CORRESPONDING DUALITY GAP.

### D. Computation Results

The case study has been run considering exactly the same operating conditions as in [9], with the line outage occurring at time $t = 100\,$s. The setup features a 2.8 GHz Pentium running Matlab R13 and CPLEX 8.0 as MILP solver, and solutions have been obtained and compared for both the Lagrangian scheme and the full problem. These are reported in Table 1, where the indicated time represents the required computation time for a single sampling interval. The duality gap $\gamma$ in percentage terms of the Lagrangian scheme is calculated as $\gamma = 100(J^+ - \bar{Z})/\bar{Z}$, in which $\bar{Z}$ corresponds to the value of the dual evaluated at $\bar{\lambda}$. As can be seen, the temporal Lagrangian decomposition scheme yields computation times which are improved by an order of magnitude. In particular, the maximum computation time is reduced by a factor of 20 relative to the full problem. Most important, it is less than the sampling interval allowing for an on-line implementation of the control scheme. Furthermore, the duality gap lies within a range of $10\%$.

It is worthwhile noting that for this model and for the featured sequence of operating points the suboptimal $J^+$ actually coincides with $J^*$, i.e. the optimum is in actual fact obtained. This is partly due to the fact that the duality gap is small and that the inputs are purely binary preventing small deviations in the input variables that would otherwise occur. Although not strictly representative of the intended purpose of the presented algorithm, this result is consistent with the fact that the derived control sequence is indeed in a region reasonably close to optimality.

For the case study considered, the control problem may thus be tackled with a substantially lower computational effort, so that the effective potential for future applications appears to be encouraging.

## V. Conclusions

Lagrangian decomposition has been extensively applied to a variety of optimization problems and as such represents a consolidated technique in the ambit of mathematical programming. In this paper, its domain of application has been further extended by devising a workable and effective temporal decomposition method that can be directly applied to the standard MPC problem formulation for MLD systems. The computational results reported for the hybrid case study confirm that the proposed scheme yields computation times reduced by an order of magnitude and produces reasonably accurate numerical values compared with the original full optimization problem.

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## References

[1] A. Bemporad, F. Borrelli, and M. Morari. Piecewise linear optimal controllers for hybrid systems. In *Proc. American Control Conf.*, pages 1190–1194, Chicago, IL, June 2000.

[2] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics and constraints. *Automatica*, 35(3):407–427, March 1999.

[3] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, January 2002.

[4] D. P. Bertsekas. *Non Linear Programming*. Athena Scientific, 1995.

[5] P. J. Campo and M. Morari. Robust model predictive control. In *Proc. American Control Conf.*, volume 2, 1987.

[6] M. L. Fisher. The Lagrangian relaxation method for solving integer programming problems. *Management Science*, 27(2):1–18, January 1982.

[7] M. L. Fisher. An applications oriented guide to Lagrangian relaxation. *Interfaces*, 15(2):10–21, March-April 1985.

[8] T. Geyer, M. Larsson, and M. Morari. Hybrid control of voltage collapse in power systems. Technical Report AUT02-12, Automatic Control Laboratory ETH Zurich, `http://control.ee.ethz.ch`, 2002.

[9] T. Geyer, M. Larsson, and M. Morari. Hybrid emergency voltage control in power systems. In *Proc. of the European Control Conf.*, Cambridge, UK, September 2003.

[10] M.H. Held, P. Wolfe, and H.D. Crowder. Validation of subgradient optimization. *Mathematical Programming Study*, 6(1):62–88, 1974.

[11] J. R. Jackson and I. E. Grossmann. Temporal decomposition scheme for nonlinear multisite production planning and distribution models. *Industrial and Engineering Chemistry Research*, 43:3045–3055, 2003.

[12] P.O.M. Scokaert, D.Q. Mayne, and J.B. Rawlings. Suboptimal model predictive control (feasibility implies stability). *IEEE Trans. Automat. Contr.*, 44(3):648–654, 1999.